



TITLE:

Sperical Sections of a Homogeneous Vector Bundle (群の 表現と調和解析)

AUTHOR(S):

MINEMURA, KATSUHIRO

CITATION:

MINEMURA, KATSUHIRO. Sperical Sections of a Homogeneous Vector Bundle (群の表現と調和解析). 数理解析研究所講究録 1979, 368: 33-37

ISSUE DATE:

1979-11

URL:

<http://hdl.handle.net/2433/104644>

RIGHT:

Spherical Sections of a Homogeneous Vector Bundle

Katsuhiro Minemura

In this note, we show some results on the integral representation of eigensections of invariant differential operators on a homogeneous vector bundle over a riemannian symmetric space. First we determine the structure of the algebra \mathcal{D} of invariant differential operators on a homogeneous vector bundle, define the notions of eigensections and spherical sections corresponding to a given finite-dimensional representation of the algebra \mathcal{D} and obtain the dimension formula and an integral representation of spherical sections. The notion of spherical sections is a generalization of that of zonal spherical functions. The dimension formula of the space of spherical sections will be crucial for the problem of Poisson integral representation of eigensections, which I would like to solve during this summer.

Precisely speaking, let G be a connected real semisimple Lie group with Lie algebra \mathfrak{g} of finite center, K a maximal compact subgroup with Lie algebra \mathfrak{k} , $G = KAN$ an Iwasawa decomposition with split torus A and $g = K(g)e^{H(g)}n(g)$ ($g \in G$, $K(g) \in K$, $e^{H(g)} \in A$, $n(g) \in N$) the decomposition of g in G corresponding to $G = KAN$, where $H(g)$ is an element in the

Lie algebra \mathcal{A} of A . Let E denote the homogeneous vector bundle over G/K associated to a given irreducible unitary representation τ of K and \mathbb{D}_τ denote the algebra of invariant differential operators on E . Let \mathfrak{g} (resp. \mathfrak{k}) denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ (resp. $\mathfrak{k}_\mathbb{C}$) and $*$ denote the anti-automorphism of \mathfrak{g} defined by $X^* = -X$ ($X \in \mathfrak{g}$). Let \mathfrak{g}^K be the centralizer of K in \mathfrak{g} , \mathcal{I}_τ the kernel of $d\tau$ in \mathfrak{k} and \mathcal{I}_τ^* the set of z^* ($z \in \mathcal{I}_\tau$). Then \mathbb{D}_τ is canonically isomorphic to the algebra $\mathfrak{g}^K / \mathfrak{g}^K \cap \mathfrak{g} \mathcal{I}_\tau^*$.

Let ρ denote the half sum of the roots corresponding to N . For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, put

$$P_{\tau, \lambda}(g) = e^{-(\lambda + \rho)H(g^{-1})} \tau(K(g^{-1})) \quad (g \in G).$$

Then there exists an algebra homomorphism, say $\chi_{\tau, \lambda}$, of \mathbb{D}_τ into $\text{End}_M(V)$ such that

$$\Delta P_{\tau, \lambda}(g) = P_{\tau, \lambda}(g) \circ \chi_{\tau, \lambda}(\Delta) \quad (\Delta \in \mathbb{D}_\tau),$$

where V denotes the representation space of τ and $\text{End}_M(V)$ denotes the set of endomorphisms on V which commute with $\tau(m)$ ($m \in M = Z_K(A)$ = the centralizer of A in K).

For an irreducible representation (σ, V_σ) of M , put $H_\sigma = \text{Hom}_M(V_\sigma, V)$ and $H_{\tau, \sigma} = \text{Hom}_M(V, V_\sigma)$. Then $\chi_{\tau, \lambda}$ defines a representation $\chi_{\tau, \sigma, \lambda}$ of \mathbb{D}_τ on H_σ by

$$\chi_{\tau, \sigma, \lambda}(\Delta)a = \chi_{\tau, \lambda}(\Delta) \circ a \quad (a \in H_\sigma).$$

Let $\mathcal{A}(E)$ denote the space of analytic sections of E . Given a

representation (χ, H) of \mathbb{D}_τ , we call $u \in \mathcal{Q}(E)$ an eigensection of type χ , if there exists a finite number of \mathbb{D}_τ -invariant subspaces H_i in $\mathcal{Q}(E)$ such that as a representation space of \mathbb{D}_τ , each H_i is isomorphic to a quotient representation of (χ, H) . Let $\mathcal{Q}(E, \chi)$ denote the space of eigensections of type χ .

Let $F_{\sigma, \lambda}$ denote the vector bundle over G/MAN associated to the representation $\sigma \otimes e^{-\lambda+\rho} \otimes 1$ of MAN on V_σ and let $\mathcal{B}(F_{\sigma, \lambda})$ denote the space of $F_{\sigma, \lambda}$ -valued hyperfunctions on G/MAN . For $\varphi \otimes a \in \mathcal{B}(F_{\sigma, \lambda}) \otimes H_\sigma$, put

$$P_{\tau, \sigma, \lambda}(\varphi \otimes a) = \int_K P_{\tau, \lambda}(k^{-1}g) a \varphi(k) dk.$$

Then it is easy to see that $P_{\tau, \sigma, \lambda}$ gives a G -linear mapping of $\mathcal{B}(F_{\sigma, \lambda}) \otimes H_\sigma$ into $\mathcal{Q}(E, \chi_{\tau, \sigma, \lambda})$.

Let $\mathcal{Q}(E, \chi_{\tau, \sigma, \lambda})^\tau$ denote the subspace consisting of the sections in $\mathcal{Q}(E, \chi)$ which transform according to τ under the left regular representation of K on $\mathcal{Q}(E)$. We identify $V \otimes H_{\tau, \sigma}$ with the subspace of sections in $\mathcal{B}(F_{\sigma, \lambda})$ which transform according to τ under the left regular representation of K on $\mathcal{B}(F_{\sigma, \lambda})$ by

$$V \otimes H_{\tau, \sigma} \ni v \otimes b \mapsto \varphi_{v, b}(k) = b\tau(k^{-1})v \in \mathcal{B}(F_{\sigma, \lambda})$$

$$(v \in V, b \in H_{\tau, \sigma}, k \in K),$$

where $\varphi_{v, b}$ is regarded as an element in $\mathcal{B}(F_{\sigma, \lambda})$ by

$$\varphi_{v, b}(kan) = e^{(\lambda-\rho)H(a)} \varphi_{v, b}(k).$$

Then the mapping

$$\begin{aligned} v \otimes b \otimes a &\longmapsto P_{\tau, \sigma, \lambda}(\varphi_{v, b} \otimes a) \\ &= \int_K P_{\tau, \lambda}(k^{-1}g)ab\tau(k^{-1})vdk \end{aligned}$$

gives a linear mapping of $V \otimes H_{\tau, \sigma} \otimes H_{\sigma}$ onto $\mathcal{A}(E, \chi_{\tau, \sigma, \lambda})^{\tau}$. As a corollary, using the "subrepresentation theorem" of Casselman, we have a linear isomorphism

$$\mathcal{A}(E, \chi)^{\tau} \cong V \otimes (\mathbb{D}_{\tau} / \text{Ker } \chi)^{*}$$

for any irreducible representation χ of \mathbb{D}_{τ} .

Now it is very interesting to study when $P_{\tau, \sigma, \lambda}$ will give a G -isomorphism of $\mathcal{B}(F_{\sigma, \lambda}) \otimes H_{\sigma}$ onto $\mathcal{A}(E, \chi_{\tau, \sigma, \lambda})$. We have the following conjecture:

Under a certain regularity condition on λ , we can construct a boundary value mapping $\beta_{\tau, \sigma, \lambda}$ of $\mathcal{A}(E, \chi_{\tau, \sigma, \lambda})$ into $\mathcal{B}(F_{\sigma, \lambda}) \otimes H_{\sigma}$ such that

$$\beta_{\tau, \sigma, \lambda} \circ P_{\tau, \sigma, \lambda}(\varphi \otimes a) = \varphi \otimes c(\tau, \lambda)a, \quad \varphi \in \mathcal{B}(F_{\sigma, \lambda}), \quad a \in H_{\sigma},$$

where $c(\tau, \lambda)$ is the generalized c -function.

It seems not so hard to prove it now because almost all necessary lemmas have been proved and we have only to see what differential operators in \mathbb{D}_{τ} will be necessary and suitable in order to determine the vector bundle to which the boundary values should belong.

References

- (1) S. Helgason, Invariant differential operators on homogeneous manifolds, Bull. Amer. Math. Soc. 83 (1977), 751-774.
- (2) J. Lepowsky, Algebraic results on representations of semisimple Lie groups, Trans. Amer. Math. Soc. 176 (1973), 1-44.
- (3) K. Minemura, Invariant differential operators and spherical sections of a homogeneous vector bundle, preprint.
- (4) K. Okamoto, Harmonic analysis on homogeneous vector bundles, Lecture Notes in Math., Vol. 266, pp. 255-271, Springer-Verlag, 1972.
- (5) N. R. Wallach, Representations of semisimple Lie groups and Lie algebras, Canad. Math. Congress, 1978.